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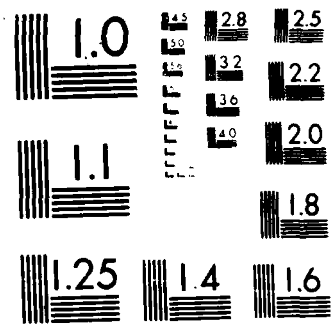

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Recursive Linear Smoothing for the 2-D Helmholtz Equation *

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ABSTRACT

A fast algorithm for reconstructing images governed by a 2-D Helmholtz equation is presented. The computational complexity of the algorithm is $O(NM \log M)$ or $O(NM^2)$ depending on boundary conditions, where N and M are the number of spatial grid points in the x and y directions respectively. This problem arises when smoothing a large number of images governed by the 2-D wave equation, because a Fourier transform in time gives a new set of images governed by the Helmholtz equation. When the images come from a scattering process, we show that a linear least-squares Born inversion of the wave field amplitudes can be performed during the smoothing procedure without changing the computational complexity. We also show that the smoothing algorithm is well-posed, and ~~that~~ the sample functions of the smoothed estimate possess smoothness properties consistent with the Helmholtz equation.

1. Introduction

In this paper we derive a fast, recursive, linear least-squares smoothing algorithm for the 2-D Helmholtz equation. Our algorithm can be used, for example, to smooth a large number of images governed by the 2-D wave equation arising in acoustical holography [8] or in oceanic surveillance. If we assume that the vibrating system is in steady state, and that the inputs and observation noise are temporally stationary, then a

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Fourier expansion with respect to time gives a new set of images, indexed by the frequency f, that are governed by the Helmholtz equation whose wavenumber k varies with f. Each transformed image can be smoothed separately, and estimates of the original time-domain images can then be obtained by an inverse Fourier transform.

In order to smooth images governed by the Helmholtz equation, we reformulate the equation as a well-posed, distributed-parameter, acausal linear system, and use the recent extension of Adams, Willsky and Levy [1] of the method of complementary models [9] to write the Hamiltonian system for the smoothed estimate. Transforming in one direction produces a set of indexed, well-posed, finite-dimensional, acausal linear systems which we solve recursively using a diagonalizing change of variables. The complexity of our algorithm for each wavenumber k is $O(NM\log M)$ or $O(NM^2)$ depending on the boundary conditions, where N is the number of grid points in the x direction and M is the number in the (orthogonal) y direction.

Another approach to the smoothing of images governed by the Helmholtz equation involves the use of the Karhunen-Loeve expansion of the wave field [2],[3],[4], resulting in an algorithm with complexity $O(MN\log MN)$. However, the boundary conditions are required to be conservative and have no random inputs.

In contrast, we can handle x-boundary conditions that are conservative or dissipative and that include random inputs. We note that work using a similar approach is discussed by Yoshida and Ogura in [10]. In their work, the dynamics are discrete and the underlying random field is homogeneous (spatially stationary), whereas in this paper, the dynamics are continuous and the random field is not required to be homogeneous. Furthermore, an important step in the derivation of the estimator in [10] is the replacement of a non-Markovian random process with a Markovian random process having the same mean and covariance. In our approach, this realization step is not needed.

2. Problem Statement

Consider the scalar Helmholtz equation on the rectangle $\Omega = [0,L_1] \times [0,L_2]$:

$$u_{xx}+u_{yy}+k^2 u = \epsilon(x,y) \tag{2.1}$$

$$u = u(x,y) \ , \ (x,y) \in \Omega$$

with boundary conditions

$$u(0,y) = v_1(y) \ , \ u(L_1,y) = v_2(y) \tag{2.2a}$$

$$u(x,0) = 0 \ , \ u(x,L_2) = 0 \ . \tag{2.2b}$$

Here

$$k^2 = k_0^2 + j\eta \ , \ k_0 = \frac{2\pi f}{c} \ , \ \eta > 0$$

ϵ is the input field, v_1 and v_2 are boundary inputs on the x-axis, u is the wavefield amplitude, f is the temporal

frequency of interest, c is the phase velocity of the medium, and η is the damping term. The observations are

$$z(x,y) = u(x,y)+w(x,y) \ , \ (x,y) \in \Omega \tag{2.3}$$

where $w(x,y)$ is the observation noise field.

We shall make the following statistical assumptions: (1) the driving field ϵ and observation noise field w are zero mean and white with constant intensities q and r, respectively, and are uncorrelated with each other, (2) if $v(y) = [v_1(y) \ \ v_2(y)]'$ then

$$Ev(y) = 0$$

$$Ev(y)v(s)' = \Pi_v \delta(y-s)$$

where Π_v is invertible and v is uncorrelated with ϵ and w.

The estimation problem of interest here is to determine the linear least-squares estimate $\hat{u}(x,y)$ of $u(x,y)$, $(x,y) \in \Omega$, given the observations (2.3) over the entire rectangle Ω .

To see how the Helmholtz equation may arise in practice, consider the 2-D wave equation on the rectangle $\Omega = [0,L_1] \times [0,L_2]$:

$$u_{tt} - c^2(u_{xx}+u_{yy})+\gamma u_t = d(x,y,t)$$

$$u = u(x,y,t) \ , \ (x,y,t) \in \Omega \times [T_0,T_1]$$

with boundary conditions

$$u(0,y,t) = v_1(y,t)\ ,\ u(L_1,y,t) = v_2(y,t)$$

$$u(x,0,t) = 0\ ,\ u(x,L_2,t) = 0$$

$$u(x,y,T_0) = u_t(x,y,T_0) = 0$$

and observations

$$z(x,y,t) = u(x,y,t)+w(x,y,t)$$

Let $T_0 \rightarrow -\infty$ and assume that the observation interval is $[0,T_1]$ where T_1 is very large. Then a Fourier series expansion of the observations will give a new set of images $z(x,y,f)$ that are governed by (2.1)-(2.3), where the dependence on f has been suppressed, and where $\epsilon = -d/c^2$ and $\eta = 2\pi f \gamma$. We assume that the input field $d(x,y,t)$, boundary inputs $v_1(y,t)$, $v_2(y,t)$, and observation noise $w(x,y,t)$ are wide-sense stationary in time. The stationarity assumption implies that for $f_1 \neq f_2$, $z(x,y,f_1)$ and $u(x,y,f_1)$ are uncorrelated with $z(x,y,f_2)$, since $T_1 \rightarrow \infty$. We can therefore solve an uncoupled set of smoothing problems for the 2-D Helmholtz equation (indexed by f), and then inverse transform to recover the time-domain estimates.

3. State Space Formulation and Characterization of the Estimate

In order to put (2.1)-(2.3) in state-variable form, define an operator T with domain $D(T)$ as follows:

$$Tf = -(k^2 f(x,y) + f_{yy}(x,y)),\ f \in D(T)$$

where

$$D(T)=\{f \in \mathbf{L}_2(\Omega):f,f_y \ abs.\ cont.,\ f_{yy} \in \mathbf{L}_2(\Omega),$$
$$f(x,0) = f(x,L_2) = 0\}$$

Also define the state vector $m(x,y)$ as

$$m(x,y) = [m_1(x,y),m_2(x,y)]' = [u(x,y),u_x(x,y)]'$$

We can now rewrite (2.1)-(2.3) as

$$\frac{\partial}{\partial x}m(x,y) = Am(x,y)+B\epsilon(x,y) \tag{3.1a}$$

$$m_1 \in D(T)$$

$$v(y) = V_0 m(0,y)+V_L m(L_1,y) \tag{3.1b}$$

$$z(x,y) = Cm(x,y)+w(x,y) \tag{3.1c}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ T & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$V_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ V_L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Equations (3.1) are in the form of a distributed parameter, acausal linear system. Finite-dimensional acausal linear systems are discussed in [6]-[7]. We show in Appendix A that (3.1) are well posed, in the sense that if $\epsilon = v = 0$, then $m = 0$.

Using results in [1], we can express the linear least-squares estimate $\hat{m}$ of m, given the observations (2.3), as the solution of the following Hamiltonian system:

$$\frac{\partial}{\partial x}\begin{bmatrix} \hat{m}(x,y) \\ \hat{\lambda}(x,y) \end{bmatrix} = \begin{bmatrix} A & qBB' \\ r^{-1}C'C & -A^* \end{bmatrix}\begin{bmatrix} \hat{m}(x,y) \\ \hat{\lambda}(x,y) \end{bmatrix} + \begin{bmatrix} 0 \\ -C'r^{-1}z(x,y) \end{bmatrix} \tag{3.2a}$$

$$\hat{m} \in D(A)\ ,\ \hat{\lambda} \in D(A^*)$$

$$0 = V^*\Pi_v^{-1} V \begin{bmatrix} \hat{m}(0,y) \\ \hat{m}(L_1,y) \end{bmatrix} + \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\lambda}(0,y) \\ \hat{\lambda}(L_1,y) \end{bmatrix} \qquad (3.2b)$$

where

$$V = [V_0 \ \ V_L]$$

the asterisk denotes adjoint, and $D(A)$, $D(A^*)$ denote the domains of A and A^*. These domains are

$$D(A) = \{[f_1, f_2]' : f_1 \in D(T)\ ,\ f_2 \in \mathbf{L}_2(\Omega)\}$$

$$D(A^*) = \{[f_1, f_2]' : f_1 \in \mathbf{L}_2(\Omega)\ ,\ f_2 \in D(T)\}$$

Moreover, the input estimate satisfies

$$\hat{\epsilon}(x,y) = qB^*\hat{\lambda}(x,y) \qquad (3.2c)$$

Eq. (3.2c) can be interpreted as a linear least-squares Born inversion when the observations are the scattered wavefield in an inverse scattering experiment.

Instead of solving (3.2) directly for $\hat{m}$ and $\hat{\lambda}$, we will transform (3.2) with respect to y using the discrete sine transform $\mathbf{S}$, given by

$$\mathbf{S}g = \frac{1}{L_2}\int_0^{L_2} \sin(py)\, g(y)\, dy$$

$$p = \frac{2\pi n}{L_2}\ ,\ n = 0, \pm 1, \pm 2, \cdots$$

It can easily be verified that $\mathbf{S}T = (p^2 - k^2)\mathbf{S}$ and thus $\mathbf{S}A = A_p\mathbf{S}$ where

$$A_p = \begin{bmatrix} 0 & 1 \\ p^2 - k^2 & 0 \end{bmatrix}.$$

Transforming (3.2a) with respect to y gives

$$\frac{\partial}{\partial x}\begin{bmatrix}\hat{m}(x,p)\\ \hat{\lambda}(x,p)\end{bmatrix} = \begin{bmatrix}A_p & qBB'\\ r^{-1}C'C & -A_p^*\end{bmatrix}\begin{bmatrix}\hat{m}(x,p)\\ \hat{\lambda}(x,p)\end{bmatrix} + \begin{bmatrix}0\\ -C'r^{-1}z(x,p)\end{bmatrix} \qquad (3.3a)$$

where $\hat{m}(x,p) = \mathbf{S}\hat{m}(x,y)$,etc . Transforming (3.2b) with respect to y gives

$$0 = V^*\Pi_v^{-1}V\begin{bmatrix}\hat{m}(0,p)\\ \hat{m}(L_1,p)\end{bmatrix} + \begin{bmatrix}-I & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\hat{\lambda}(0,p)\\ \hat{\lambda}(L_1,p)\end{bmatrix}.$$

When written in the standard form for an acausal linear system, these boundary conditions are

$$0 = W_0\begin{bmatrix}\hat{m}(0,p)\\ \hat{\lambda}(0,p)\end{bmatrix} + W_L\begin{bmatrix}\hat{m}(L_1,p)\\ \hat{\lambda}(L_1,p)\end{bmatrix} \qquad (3.3b)$$

where

$$W_0 = \begin{bmatrix}V_0^*\Pi_v^{-1}V_0 & -I\\ V_L^*\Pi_v^{-1}V_0 & 0\end{bmatrix} \text{ and } W_L = \begin{bmatrix}V_0^*\Pi_v^{-1}V_L & 0\\ V_L^*\Pi_v^{-1}V_L & I\end{bmatrix}$$

We see that in the p domain the estimate Hamiltonian is decoupled; that is, (3.3) are simply indexed by p, and can be solved individually. In the next section we show how to solve (3.3) recursively for each p. Before doing so, we first consider whether (3.3) is well-posed. These equations can be shown to be well-posed by first realizing that for each p, (3.3) is the estimate Hamiltonian of another acausal linear system, the so-called p-dynamics of the Helmholtz equation.

$$\frac{\partial}{\partial x}m(x,p) = A_p m(x,p) + B\epsilon(x,p) \qquad (3.4a)$$

$$V_0 m(0,p) + V_L m(L_1,p) = v(p) \qquad (3.4b)$$

$$z(x,p) = Cm(x,p)+w(x,p) \tag{3.4c}$$

where we have transformed (3.1) into the p-domain by S. The invertibility of $V_0 + V_L e^{A_p L_1}$ is necessary and sufficient for (3.4) to be well-posed [6]-[7]. It is easy to verify that this matrix is invertible for all p. To prove then that (3.3) is well-posed for all p, assume that the input $z(x,p)$ is identically zero . Now $\hat{m}(x,p)$ is the linear least-squares estimate of $m(x,p)$ based on observations which are identically zero, so $\hat{m}(x,p) = Em(x,p) = 0$, the last equality following from the well-posedness of (3.1). As a result $\hat{\lambda}(x,p)$ satisfies

$$\frac{\partial}{\partial x}\hat{\lambda}(x,p) = -A_p^*\hat{\lambda}(x,p)$$

and

$$\hat{\lambda}(0,p) = \hat{\lambda}(L_1,p) = 0$$

and thus $\hat{\lambda}(x,p) = 0$.

4. Recursive Solution of the Estimate Hamiltonian

We shall derive our recursive smoothing formula by diagonalizing the dynamics in (3.3a) via a change of variables. Let

$$H_p = \begin{bmatrix} A_p & qBB' \\ r^{-1}C'C & -A_p^* \end{bmatrix}$$

The characteristic equation for H_p is given by

$$\lambda^4 - 2\mathrm{Re}(\alpha)\lambda^2 + |\alpha|^2 + r^{-1}q = 0$$

where $\alpha = p^2 - k^2$. Solutions to this equation are λ_0 , $\overline{\lambda_0}$, $-\lambda_0$, $-\overline{\lambda_0}$, where

$$\lambda_0 = \left(\frac{(|\alpha|^2 + r^{-1}q)^{1/2} + \mathrm{Re}\alpha}{2}\right)^{1/2} - j\left(\frac{(|\alpha|^2 + r^{-1}q)^{1/2} - \mathrm{Re}\alpha}{2}\right)^{1/2} \quad (4.1)$$

The four eigenvalues of H_p are all distinct because both the real and imaginary parts of (4.1) are non-zero for all p. As a result, we can diagonalize H_p as follows :

$$M_p^{-1} H_p M_p = \begin{bmatrix} \Lambda_p & 0 \\ 0 & -\Lambda_p \end{bmatrix}$$

where

$$\Lambda_p = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \overline{\lambda_0} \end{bmatrix}$$

and the modal matrix M_p and its inverse are given explicitly by

$$M_p = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_0 & \overline{\lambda_0} & -\lambda_0 & -\overline{\lambda_0} \\ -\lambda_0 c & -\overline{\lambda_0} d & \lambda_0 c & \overline{\lambda_0} d \\ c & d & c & d \end{bmatrix}$$

$$M_p^{-1} = \frac{jq}{4\sigma} \begin{bmatrix} d & d\lambda_0^{-1} & \lambda_0^{-1} & -1 \\ -c & -c\overline{\lambda_0}^{-1} & -\overline{\lambda_0}^{-1} & 1 \\ d & -d\lambda_0^{-1} & -\lambda_0^{-1} & -1 \\ -c & c\overline{\lambda_0}^{-1} & \overline{\lambda_0}^{-1} & 1 \end{bmatrix}$$

with

$$c = j(\eta + \sigma)/q$$

$$d = j(\eta - \sigma)/q$$

$$\sigma = (qr^{-1} + \eta^2)^{1/2}$$

Now by using the following change of variables in (3.3):

$$\begin{bmatrix} \Psi_f(x,p) \\ \Psi_b(x,p) \end{bmatrix} = M_p^{-1} \begin{bmatrix} \hat{m}(x,p) \\ \hat{\lambda}(x,p) \end{bmatrix} \tag{4.2}$$

we get

$$\frac{\partial}{\partial x} \begin{bmatrix} \Psi_f \\ \Psi_b \end{bmatrix} = \begin{bmatrix} \Lambda_p & 0 \\ 0 & -\Lambda_p \end{bmatrix} \begin{bmatrix} \Psi_f \\ \Psi_p \end{bmatrix} + \begin{bmatrix} B_f \\ B_b \end{bmatrix} z \tag{4.3a}$$

$$0 = \begin{bmatrix} V_f^0 & V_b^0 \end{bmatrix} \begin{bmatrix} \Psi_f(0,p) \\ \Psi_b(0,p) \end{bmatrix} + \begin{bmatrix} V_f^L & V_b^L \end{bmatrix} \begin{bmatrix} \Psi_f(L_1,p) \\ \Psi_b(L_1,p) \end{bmatrix} \tag{4.3b}$$

where

$$\begin{bmatrix} V_f^0 & V_b^0 \end{bmatrix} = W_0 M_p \ , \ \begin{bmatrix} V_f^L & V_b^L \end{bmatrix} = W_L M_p$$

$$\begin{bmatrix} B_f \\ B_b \end{bmatrix} = M_p^{-1} \begin{bmatrix} 0 \\ -C'r^{-1} \end{bmatrix}$$

Equations (4.3) are in a form which can be solved recursively. In terms of $\Psi_f(0,p)$and $\Psi_b(L_1,p)$, a solution to (4.3a) is

$$\Psi_f(x,p) = e^{\Lambda_p(x)} \Psi_f(0,p) + \Psi_f^0(x,p) \tag{4.4a}$$

$$\Psi_b(x,p) = e^{\Lambda_p(L_1 - x)} \Psi_b(L_1,p) + \Psi_b^0(x,p) \tag{4.4b}$$

where

$$\frac{\partial}{\partial x} \Psi_f^0(x,p) = \Lambda_p \Psi_f^0(x,p) + B_f z(x,p) \tag{4.5a}$$

$$\frac{\partial}{\partial x} \Psi_b^0(x,p) = -\Lambda_p \Psi_b^0(x,p) + B_b z(x,p) \tag{4.5b}$$

$$\Psi_f^0(0,p) = 0 \ , \ \Psi_b^0(L_1,p) = 0. \tag{4.5c}$$

Note that (4.5a) and (4.5b) are stable in the forward and backward directions, respectively. Setting $x = L_1$ in

(4.4a), and $x = 0$ in (4.4b), we can solve for $\Psi_f(0,p)$ and $\Psi_b(L_1,p)$ in (4.3b) as

$$\begin{bmatrix} \Psi_f(0,p) \\ \Psi_b(L_1,p) \end{bmatrix} = -F_{fb}^{-1}\{V_f^L \Psi_f^0(L_1,p) + V_b^0 \Psi_b^0(0,p)\}$$

where

$$F_{fb} = \left[V_f^0 + V_f^L e^{\Lambda_+ L_1} \vdots V_b^L + V_b^0 e^{\Lambda_- L_1} \right]$$

A solution to (4.3) is therefore given by

$$\begin{bmatrix} \Psi_f(x,p) \\ \Psi_b(x,p) \end{bmatrix} = \begin{bmatrix} -e^{\Lambda_+ x} & 0 \\ 0 & -e^{\Lambda_-(L_1 - x)} \end{bmatrix} F_{fb}^{-1}\{V_f^L \Psi_f^0(L_1,p) + V_b^0 \Psi_b^0(0,p)\}$$

$$+ \begin{bmatrix} \Psi_f^0(x,p) \\ \Psi_b^0(x,p) \end{bmatrix} \tag{4.6}$$

The F_{fb} matrix is invertible because

$$F_{fb} = F_H M_p \begin{bmatrix} I & 0 \\ 0 & e^{\Lambda_- L_1} \end{bmatrix},$$

where F_H is the invertible matrix associated with (3.3) being well-posed:

$$F_H = W_0 + W_L e^{H_+ L_1}.$$

The behavior of the algorithm as $p \to \infty$ needs to be examined further. We will show that the determinant of F_{fb} does not vanish as $p \to \infty$. As p gets large, one can ignore the exponential terms in F_{fb} and write

$$F_{fb} \approx \begin{bmatrix} \theta_{11} + \lambda_0 c & \theta_{11} + \bar{\lambda}_0 d & \theta_{12} & \theta_{12} \\ -c & -d & 0 & 0 \\ \theta_{21} & \theta_{21} & \theta_{22} + \lambda_0 c & \theta_{22} + \bar{\lambda}_0 d \\ 0 & 0 & c & d \end{bmatrix}$$

where θ_{ij} is the ij-th entry of Π_v^{-1}. Then

$$\det F_{fb} \approx -(c-d)^2\det(\Pi_v^{-1}) - c^2d^2(\lambda_0-\bar{\lambda}_0)^2 + cd(c-d)(\lambda_0-\bar{\lambda}_0)(\theta_{11}+\theta_{22})$$

As $p \to \infty$, $\lambda_0-\bar{\lambda}_0 \to 0$, so that

$$\lim_{p\to\infty} \det F_{fb} = 4(qr^{-1}+\eta^2)q^{-2}\det(\Pi_v^{-1}) \neq 0$$

It is shown in Appendix B that under realistic energy assumptions on the observed images, $\Psi_f(x,p)$ and $\Psi_b(x,p)$ decay fast enough as $p\to\infty$ to ensure that $\hat{m}(x,y) \in D(A)$ and $\hat{\lambda} \in D(A^*)$.

To summarize, the solution procedure for solving the estimator equations (3.2) is (1) transform the observations into the p-domain, (2) compute $\Psi_f^0(x,p)$ and $\Psi_b^0(x,p)$ using (4.5), (3) find $\Psi_f(x,p)$ and $\Psi_b(x,p)$ from (4.6), (4) compute $\hat{m}(x,p)$ using (4.2), (5) inverse transform $\hat{m}(x,p)$ to get $\hat{m}(x,y)$.

5. Other Boundary Conditions

The smoothing problem for the 2-D Helmholtz equation with more general boundary conditions can be handled in a manner very similar to that just described. In addition, a wavenumber that varies in the y direction can be studied. The boundary conditions we consider in this section are conservative on the y-boundaries, and, in general, dissipative on the x-boundaries. To derive the estimator for such problems, we start off with the same 2-D Helmholtz equation (2.1), and x-boundary conditions (2.2a). The y-boundary conditions are now

$$\cos\vartheta\, u(x,0)+\sin\vartheta\, u_y(x,0) = 0 \qquad (5.1a)$$

$$\cos\gamma \; u(x,L_2)+\sin\gamma \; u_y(x,L_2) = 0 \tag{5.1b}$$

where β and γ are real. Periodic boundary conditions

$$u(x,0) = u(x,L_2) \;,\; u_y(x,0) = u_y(x,L_2) \tag{5.2}$$

could also be assumed. The wavenumber k satisfies

$$k^2(y) = k_0^2(y)+j\eta,\; \eta>0$$

The estimate Hamiltonian for these boundary conditions and wavenumber is the same as (3.2), however the operator T is defined as

$$Tf = -(\partial_{yy}+k^2(y))f$$

with domain

$$D(T) = \{f \in \mathbf{L}_2(\Omega):f, f_y \; abs.\; cont.,\; f_{yy} \in \mathbf{L}_2(\Omega),$$
$$f \; satisfies \; (5.1) \; or \; (5.2)\}$$

We now introduce the selfadjoint Sturm-Liouville operator

$$Qf = -(\partial_{yy}+k_0^2(y))f$$

with domain

$$D(Q) = D(T)$$

The operator Q has a countably infinite number of eigenvalues μ_p and eigenfunctions $\Phi_p(y)$. If we use the transform operator $\mathbf{K}$ defined by

$$(\mathbf{K}f)_p = \frac{1}{c_p}\int_0^{L_2} \Phi_p(y)f(y)\,dy$$

$$c_p^2 = \int_0^{L_2} \Phi_p^2(y)\,dy$$

then $\mathbf{K}A = A_p\mathbf{K}$, where

$$A_p = \begin{bmatrix} 0 & 1 \\ \mu_p - j\eta & 0 \end{bmatrix}$$

From this point on, the calculations are identical to the zero-boundary condition case, except that $\mathbf{K}$ replaces $\mathbf{S}$.

To consider problems where the x boundary conditions are more general than (2.2a), we assume that the matrices V_0 and V_L are now linear operators $\mathbf{V}_0$ and $\mathbf{V}_L$ such that under the transform $\mathbf{K}$ described above, the action of the operators $\mathbf{V}_0$ and $\mathbf{V}_L$ are multiplicative, that is, the following transform relations hold

$$\mathbf{V}_0 m(0,y) \Longleftrightarrow V_0(p)m(0,p)$$

$$\mathbf{V}_L m(L_1,y) \Longleftrightarrow V_L(p)m(L_1,p)$$

where $V_0(p)$ and $V_L(p)$ are complex valued 2×2 matrix functions of p. As before, it is necessary that $V_0(p)+V_L(p)e^{A_pL}$ be invertible for all p. An example of a dissipative boundary condition occurs when

$$V_0(p) = \begin{bmatrix} s & 1 \\ 0 & 0 \end{bmatrix}, \; V_L(p) = \begin{bmatrix} 0 & 0 \\ s & -1 \end{bmatrix}$$

and s is complex. This case corresponds to a damped, elastically-braced membrane on the $x = 0$ and $x = L_1$ sides. The determinant of $V_0+V_L e^{A_pL_1}$ is

$$\left(\frac{s^2}{\sqrt{k^2-p^2}}+\sqrt{k^2-p^2}\right)\sin(\sqrt{k^2-p^2}L)+2s\cos(\sqrt{k^2-p^2}L) \qquad (5.5)$$

Typically (5.5) is non-zero.

6. Complexity Analysis

For computational purposes the rectangle Ω is discretized into an $N \times M$ grid. The complexity of the principal steps needed to calculate the least-squares estimate of the wave amplitude at one wavenumber k is given in Table 1.

TABLE 1 The Smoothing Complexity for the 2-D Helmholtz Equation

Step	Complexity: zero or periodic y-bound conds	Complexity: other y-b.c.
Find $\Phi_p(y)$ and μ_p for (5.3),(5.4)	n/a	$O(M^2 b)$
Fourier transform the observations	$O(NM \log M)$	$O(NM^2)$
Find $\hat{m}(x,p)$ via (4.5),(4.6),(4.2)	$O(NM)$	$O(NM)$
Inverse transform $\hat{m}(x,p)$	$O(NM \log M)$	$O(NM^2)$

In Table 1, b is the bandwidth of the matrix used to approximate the operator Q in the eigenvalue calculations. Typically $b = 1$. The overall complexity for each wavenumber is then either $O(NM \log M)$ or $O(NM^2)$, depending on the y-boundary conditions. Obviously if one does not exploit the particular structure of our model, and uses only first and second moment information, the complexity would be $O(M^3 N^3)$. As discussed in the Introduction, for zero boundary conditions a com-

plexity of $O(MN\log MN)$ has been achieved in [2],[3],[4] using different techniques. The complexity of transforming the time domain images into the frequency domain for T wavenumbers is $O(TNM\log T)$. Therefore, the entire smoothing procedure would have a complexity of $O(TNM\log TM)$ or $O(TNM\log T+TNM^2)$ depending on the y-boundary conditions and assuming that $b=1$.

7. The Smoothing Error Covariance

Using results in [1], we can express the smoothing error $\tilde{m}(x,y) = m(x,y) - \hat{m}(x,y)$ as the solution of the Hamiltonian system

$$\frac{\partial}{\partial x}\begin{bmatrix}\tilde{m}(x,y)\\ -\hat{\lambda}(x,y)\end{bmatrix} = \begin{bmatrix}A & qBB'\\ r^{-1}C'C & -A^*\end{bmatrix}\begin{bmatrix}\tilde{m}(x,y)\\ -\hat{\lambda}(x,y)\end{bmatrix} + \begin{bmatrix}B & 0\\ 0 & C'r^{-1}\end{bmatrix}\begin{bmatrix}\epsilon(x,y)\\ w(x,y)\end{bmatrix} \quad (7.1a)$$

$$\tilde{m} \in D(A)\ ,\ \hat{\lambda} \in D(A^*)$$

with boundary conditions (compare with (3.2))

$$V^*\Pi_v^{-1}v(y) = V^*\Pi_v^{-1}V\begin{bmatrix}\tilde{m}(0,y)\\ \tilde{m}(L_1,y)\end{bmatrix} + \begin{bmatrix}-I & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}-\hat{\lambda}(0,y)\\ -\hat{\lambda}(L_1,y)\end{bmatrix} \quad (7.1b)$$

Transforming (7.1) with respect to y gives (see Section 3)

$$\frac{\partial}{\partial x}\begin{bmatrix}m(x,p)\\ -\lambda(x,p)\end{bmatrix} = \begin{bmatrix}A_p & qBB'\\ r^{-1}C'C & -A_p^*\end{bmatrix}\begin{bmatrix}\tilde{m}(x,p)\\ -\hat{\lambda}(x,p)\end{bmatrix} + \begin{bmatrix}B & 0\\ 0 & C'r^{-1}\end{bmatrix}\begin{bmatrix}\epsilon(x,p)\\ w(x,p)\end{bmatrix} \quad (7.2a)$$

$$V^*\Pi_v^{-1}v(p) = W_0\begin{bmatrix}\tilde{m}(0,p)\\ -\hat{\lambda}(0,p)\end{bmatrix} + W_L\begin{bmatrix}\tilde{m}(L_1,p)\\ -\hat{\lambda}(L_1,p)\end{bmatrix} \quad (7.2b)$$

Our original statistical assumptions and the properties of

the transform imply that $\epsilon(x,p)$, $\epsilon(x,q)$, $w(x,p)$, $w(x,q)$, $v(p)$, $v(q)$ are mutually uncorrelated for $p \neq q$. Therefore, if we let

$$P(x,y) = E[\tilde{m}(x,y)\tilde{m}^*(x,y)]$$

$$P(x,p) = E[\tilde{m}(x,p)\tilde{m}^*(x,p)]$$

then

$$P(x,y) = \sum_{l=-\infty}^{\infty} P(x,p) \sin^2 py \ , \ p = \frac{2\pi l}{L_2}$$

Using a Green's function solution to the acausal linear system (7.2), one can show that $P(x,p)$ satisfies

$$P(x,p) = [I \vdots 0] \Delta(x,p) \begin{bmatrix} I \\ \cdots \\ 0 \end{bmatrix}$$

where

$$\Delta(x,p) = \int_0^{L_1} G(x,\sigma,p) \begin{bmatrix} qBB' & 0 \\ 0 & r^{-1}C'C \end{bmatrix} G^*(x,\sigma,p)\, d\sigma$$

$$+ e^{H_p x} F_H^{-1} V^* \Pi_v^{-1} V (F_H^*)^{-1} e^{H_p^* x} \qquad (7.3)$$

and

$$G(x,\sigma,p) = \begin{cases} e^{H_p x} F_H^{-1} W_0 e^{-H_p \sigma} & x > \sigma \\ e^{H_p x} F_H^{-1} W_L e^{H_p (L_1 - \sigma)} & x < \sigma \end{cases}$$

The overall complexity of solving these equations is $O(NM^2)$, due to the integration necessary to compute $\Delta(x,p)$ in (7.3). An alternate procedure with complexity $O(NM)$ can be derived by first diagonalizing (7.2) in a manner identical to that in Section 3.4. If we change variables using

$$\begin{bmatrix} e_f(x,p) \\ e_b(x,p) \end{bmatrix} = M_p^{-1} \begin{bmatrix} \tilde{m}(x,p) \\ -\hat{\lambda}(x,p) \end{bmatrix}$$

then

$$\frac{\partial}{\partial x} \begin{bmatrix} e_f(x,p) \\ e_b(x,p) \end{bmatrix} = \begin{bmatrix} \Lambda_p & 0 \\ 0 & -\Lambda_p \end{bmatrix} \begin{bmatrix} e_f(x,p) \\ e_b(x,p) \end{bmatrix} + \begin{bmatrix} B_f^e \\ B_b^e \end{bmatrix} \begin{bmatrix} \epsilon(x,p) \\ w(x,p) \end{bmatrix}$$

$$V^*\Pi_v^{-1} v(p) = \begin{bmatrix} V_f^0 & V_b^0 \end{bmatrix} \begin{bmatrix} e_f(0,p) \\ e_b(0,p) \end{bmatrix} + \begin{bmatrix} V_f^L & V_b^L \end{bmatrix} \begin{bmatrix} e_f(L_1,p) \\ e_b(L_1,p) \end{bmatrix}$$

where

$$\begin{bmatrix} B_f^e \\ B_b^e \end{bmatrix} = M_p^{-1} \begin{bmatrix} B & 0 \\ 0 & C' r^{-1} \end{bmatrix}$$

A solution to these equations is

$$\begin{bmatrix} e_f(x,p) \\ e_b(x,p) \end{bmatrix} = \begin{bmatrix} e^{\Lambda_p x} & 0 \\ 0 & e^{\Lambda_p (L_1 - x)} \end{bmatrix} F_{fb}^{-1} \{ V^* \Pi_v^{-1} v(p) - V_f^L e_f^0(L_1,p) - V_b^0 e_b^0(0,p) \}$$

$$+ \begin{bmatrix} e_f^0(x,p) \\ e_b^0(x,p) \end{bmatrix} \tag{7.4}$$

where

$$\frac{\partial}{\partial x} e_f^0(x,p) = \Lambda_p e_f^0(x,p) + B_f^e \begin{bmatrix} \epsilon(x,p) \\ w(x,p) \end{bmatrix} \tag{7.5a}$$

$$\frac{\partial}{\partial x} e_b^0(x,p) = -\Lambda_p e_b^0(x,p) + B_b^e \begin{bmatrix} \epsilon(x,p) \\ w(x,p) \end{bmatrix} \tag{7.5b}$$

$$e_f^0(0,p) = 0 \ , \ e_b^0(L_1,p) = 0 \tag{7.5c}$$

We can now express $\Delta(x,p)$ as

$$\Delta(x,p) = M_p \Theta(x,p) M_p^* \tag{7.6}$$

where

$$\Theta(x,p) = E \begin{bmatrix} e_f(x,p) \\ e_b(x,p) \end{bmatrix} [e_f^*(x,p) \quad e_b^*(x,p)]$$

We will express the diagonalized error covariance $\Theta(x,p)$ in terms of the second moments of the random variables $\{v, e_f^0(x), e_b^0(x)\}$, where we have suppressed the argument p for simplicity. Define the covariances

$$
\begin{aligned}
R_f(x_1,x_2) &= E[e_f^0(x_1)e_f^{0*}(x_2)] \\
R_b(x_1,x_2) &= E[e_b^0(x_1)e_b^{0*}(x_2)] \\
R_{fb}(x_1,x_2) &= E[e_f^0(x_1)e_b^{0*}(x_2)]
\end{aligned}
$$

Now $\Theta(x,p)$ can be written as

$$
\Phi_{fb}(x)F_{fb}^{-1}\{\frac{1}{2L_2}V^*\Pi_v^{-1}V + V_f^L\Pi_f(L_1)V_f^{L*} + V_f^L R_{fb}(L_1,0)V_b^{0*} +
$$

$$
V_b^0\Pi_b(0)V_b^{0*} + V_b^0 R_{fb}^*(L_1,0)V_f^{L*}\}(F_{fb}^*)^{-1}\Phi_{fb}^*(x)
$$

$$
-\Phi_{fb}(x)F_{fb}^{-1}G(x) - G^*(x)(F_{fb}^*)^{-1}\Phi_{fb}^*(x) + \begin{bmatrix}\Pi_f(x) & 0 \\ 0 & \Pi_b(x)\end{bmatrix}
$$

where

$$
\Phi_{fb}(x) = \begin{bmatrix} e^{\Lambda_p x} & 0 \\ 0 & e^{\Lambda_p(L_1-x)}\end{bmatrix}
$$

$$
G(x) = V_f^L\left[R_f(L_1,x) \vdots R_{fb}(L_1,x)\right] + V_b^0\left[R_{fb}^*(x,0) \vdots R_b(0,x)\right]
$$

$$
R_f(L_1,x) = e^{\Lambda_p(L_1-x)}\Pi_f(x) \quad , \quad R_b(0,x) = e^{\Lambda_p x}\Pi_b(x)
$$

$$
R_{fb}(x_1,x_2) = -e^{\Lambda_p(x_1-x_2)}\Pi_{fb}(x_1-x_2) \ , \ x_1 > x_2
$$

with

$$
\frac{\partial}{\partial x}\Pi_f(x) = \Lambda_p\Pi_f(x) + \Pi_f(x)\Lambda_p^* + \frac{1}{2L_2}B_f^e\begin{bmatrix} q & 0 \\ 0 & r\end{bmatrix}B_f^{e*}, \ \Pi_f(0) = 0
$$

$$
-\frac{\partial}{\partial x}\Pi_b(x) = \Lambda_p\Pi_b(x) + \Pi_b(x)\Lambda_p^* + \frac{1}{2L_2}B_b^e\begin{bmatrix} q & 0 \\ 0 & r\end{bmatrix}B_b^{e*}, \ \Pi_b(L_1) = 0
$$

$$\Pi_{fb}(x) = \frac{1}{2L_2}\int_0^x e^{-\Lambda_f u} B_f^e \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix} B_b^{e*} e^{\Lambda_f^* u}\, du$$

8. Concluding Remarks

In this paper, we have constructed a recursive smoother for the 2-D Helmholtz equation. The advantage of our method is that it leads to computationally attractive algorithms even when the x-boundary conditions are dissipative and include random inputs. Another advantage is that a linear least-squares Born inversion of the wavefield can be performed along with the image restoration without changing the computational complexity. Higher order equations characterizing vibrating plates, etc. could be approached in a manner similar to that presented here. If the wavenumber k has variations in the x-direction, then an approach based on operator Riccati equations could be used to formally diagonalize the smoother dynamics. However, the initial value problem

$$\frac{\partial}{\partial x} m(x,y) = A m(x,y)$$
$$m(0,y) = m_0(y)$$

is not well-posed (it does not lead to a semigroup), and this raises questions about the existence and uniqueness of solutions to the corresponding Riccati equations.

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Appendix A

In this appendix, we show that the state variable representation (3.1) of the Helmholtz equation is well-posed. To do this, we show that the transformed system (3.4) is well-posed, and that the solutions to these acausal linear systems give rise to Fourier coefficients that ensure that the formal Fourier series does converge and that the state vector $m(x,y) \in D(A)$.

(3.4) are well posed for every p because the matrix

$$V_0 + V_L e^{A_p L_1} = \begin{bmatrix} 1 & 0 \\ \cosh \beta L_1 & \beta^{-1}\sinh(\beta L_1) \end{bmatrix} \tag{A.1}$$

where $\beta = (p^2 - k^2)^{1/2}$, is invertible for all p. If we now change variables in (3.4) as follows:

$$\begin{bmatrix} \Upsilon_f(x,p) \\ \Upsilon_b(x,p) \end{bmatrix} = D_p^{-1} \begin{bmatrix} m_1(x,p) \\ m_2(x,p) \end{bmatrix} \tag{A.2}$$

$$D_p = \begin{bmatrix} 1 & 1 \\ -\beta & \beta \end{bmatrix}$$

then

$$\frac{\partial}{\partial x} \begin{bmatrix} \Upsilon_f(x,p) \\ \Upsilon_b(x,p) \end{bmatrix} = \begin{bmatrix} -\beta & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \Upsilon_f(x,p) \\ \Upsilon_b(x,p) \end{bmatrix} + 1/2 \begin{bmatrix} -\beta^{-1} \\ \beta^{-1} \end{bmatrix} \epsilon(x,p) \tag{A.3a}$$

$$v(p) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Upsilon_f(0,p) \\ \Upsilon_b(0,p) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \Upsilon_f(L_1,p) \\ \Upsilon_b(L_1,p) \end{bmatrix} \qquad \text{(A.3b)}$$

We can write the solution to (A.3) as

$$\begin{bmatrix} \Upsilon_f(x,p) \\ \Upsilon_b(x,p) \end{bmatrix} = (1 - e^{-2\beta L_1})^{-1} \begin{bmatrix} e^{-\beta x} & 0 \\ 0 & e^{-\beta(L_1 - x)} \end{bmatrix} \begin{bmatrix} 1 & -e^{-\beta L_1} \\ -e^{-\beta L_1} & 1 \end{bmatrix} \begin{bmatrix} v_1(p) - \Upsilon_b^0(0,p) \\ v_2(p) - \Upsilon_f^0(L_1,p) \end{bmatrix}$$

$$+ \begin{bmatrix} \Upsilon_f^0(x,p) \\ \Upsilon_b^0(x,p) \end{bmatrix} \qquad \text{(A.4)}$$

where

$$\frac{\partial}{\partial x}\Upsilon_f^0(x,p) = -\beta \Upsilon_f^0(x,p) - 1/2\ \beta^{-1}\epsilon(x,p)$$

$$\frac{\partial}{\partial x}\Upsilon_b^0(x,p) = \beta \Upsilon_b^0(x,p) + 1/2\ \beta^{-1}\epsilon(x,p)$$

$$\Upsilon_f^0(0,p) = \Upsilon_b^0(L_1,p) = 0$$

$\Upsilon_f(x,p)$ and $\Upsilon_b(x,p)$ are continuous functions of x that depend continuously on v and ϵ. Equations (3.1) will have a unique solution which depends continuously on v and ϵ if the appropriate Fourier series converge and if $m(x,y) \in D(A)$. We will assume that $(\partial^2/\partial^2 y)v(y) \in \mathbf{L}_2^2(0,L_1)$, $\epsilon(x,y) \in \mathbf{L}_2(\Omega)$ and show that this implies that $m_1(x,y) \in D(T)$ and $m_2(x,y) \in \mathbf{L}_2(\Omega)$. We first derive bounds for $\Upsilon_f^0(x,p)$ and $\Upsilon_b^0(x,p)$. Solving for $\Upsilon_f^0(x,p)$ gives

$$\Upsilon_f^0(x,p) = \frac{-1}{2\beta}\int_0^x e^{-\beta(x-u)}\epsilon(u,p)\,du$$

which gives

$$|\Upsilon_f^0(x,p)| \le \frac{1}{2|\beta|}\left[\int_0^x |e^{-\beta(x-u)}|^2 du\right]^{1/2} \left[\int_0^x |\epsilon(u,p)|^2 du\right]^{1/2}$$

where

$$\int_0^x |e^{-\beta(x-y)}|^2 dy \leq \frac{1}{2Re(\beta)}$$

Thus

$$|\Upsilon_f^0(x,p)| \leq \frac{1}{2|\beta|\sqrt{2\mathrm{Re}(\beta)}} \left[\int_0^x |\epsilon(u,p)|^2 du \right]^{1/2}$$

Since $\dot{\beta} \approx p$ when $p \to \infty$

$$|\Upsilon_f^0(x,p)|^2 \sim O(Gp^{-3})$$

where

$$G = \int_0^{L_1} |\epsilon(u,p)|^2 du$$

In a similar manner,

$$|\Upsilon_b^0(x,p)|^2 \sim O(Gp^{-3})$$

Our assumption on the input field ϵ implies that

$$\int_0^{L_2} |\epsilon(x,y)|^2 dy < \infty$$

almost everywhere with respect to x (a.e. wrt x). By Parseval's theorem

$$\sum_p |\epsilon(x,p)|^2 < \infty \quad a.e. \ wrt\ x$$

and hence by the comparison test for series

$$|\epsilon(x,p)|^2 \sim o(p^{-1})$$

a.e. wrt x . Therefore

$$|\Upsilon_f^0(x,p)|^2 \sim |\Upsilon_b^0(x,p)|^2 \sim O(p^{-4})$$

The first term in (A.4) involving the boundary term $v(p)$ is becoming exponentially small as $p \to \infty$ for all x in $(0,L_1)$. Since $(\partial^2/\partial^2 y)v(y) \in \mathbf{L}_2^2(0,L_1)$ by assumption, the first term in (A.4) has a convergent Fourier series that is

twice differentiable with respect to y on $[0,L_1]$, and is infinitely differentiable in $(0,L_1)$. The second term of (A.4) therefore dominates the solution in $(0,L_1)$ as p gets large, hence

$$|\Upsilon_f(x,p)|^2 \sim |\Upsilon_b(x,p)|^2 \sim O(p^{-4})$$

a.e. wrt x. By using the change of variables (A.1), we see that

$$|m_1(x,p)|^2 = |\Upsilon_f(x,p)+\Upsilon_b(x,p)|^2 \sim O(p^{-4})$$

and

$$|m_2(x,p)|^2 = |-\beta\Upsilon_f(x,p)+\beta\Upsilon_b(x,p)|^2 \sim O(p^{-2})$$

These bounds show that $m_1(x,y) \in D(T)$ and $m_2(x,y) \in \mathbf{L}_2(\Omega)$ [5].

Appendix B

We show in this appendix that for the 2-D Helmholtz equation, when the x-boundary conditions are separable ($\theta_{12} = \theta_{21} = 0$), and the observed image is bounded and mean-square differentiable in the y-direction,

$$\hat{m}(x,y) \in D(A) \text{ and } \hat{\lambda}(x,y) \in D(A^*)$$

To begin with, it is sufficient to show that $(\partial^2/\partial^2 y)\Psi_f(x,y)$ and $(\partial^2/\partial^2 y)\Psi_b(x,y)$ exist in the mean-square sense. The boundary conditions will be satisfied due to the sine transform.

The p-domain equations for $\Psi_{f1}{}^0, \Psi_{f2}{}^0, \Psi_{b1}{}^0, \Psi_{b2}{}^0$ are

given by ($\Psi_{fi}(x,y)$ denotes the ith component of $\Psi_f(x,y)$, etc.)

$$\Psi_{f1}{}^0(x,p) = \int_0^x e^{\lambda_0(x-s)} j\frac{q}{4\lambda_0 r\sigma} z(s,p)\,ds$$

$$\Psi_{f2}{}^0(x,p) = -\int_0^x e^{\bar{\lambda}_0(x-s)} j\frac{q}{4\bar{\lambda}_0 r\sigma} z(s,p)\,ds$$

$$\Psi_{b1}{}^0(x,p) = \int_x^{L_1} e^{-\lambda_0(x-s)} j\frac{q}{4\lambda_0 r\sigma} z(s,p)\,ds$$

$$\Psi_{b2}{}^0(x,p) = -\int_x^{L_1} e^{-\bar{\lambda}_0(x-s)} j\frac{q}{4\bar{\lambda}_0 r\sigma} z(s,p)\,ds$$

As in Appendix A, one can show that

$$\|\Psi_b^0(x,p)\|_{\mathbf{R}^2} < g_1 p^{-3/2} M(p) \qquad \text{(B.1)}$$

where

$$M^2(p) = \int_0^{L_1} |z(x,p)|^2 dx$$

and g_1 is a finite constant. The same result holds for Ψ_f^0. For separable boundary conditions [1]

$$\Psi_f(x,p) = \Psi_f^B(x,p) + \Psi_f^0(x,p)$$

Where for large p

$$\Psi_f^B(x,p) = \left[e^{\Lambda_p x} \vdots 0\right] F_{fb}^{-1} V_b^0 \Psi_b^0(0,p)$$

The $\mathbf{L}^2(\Omega)$ norm of $\Psi_f(x,y)$ can be expressed using Parseval's relation as

$$\|\Psi_f(x,y)\|^2_{\mathbf{L}^2(\Omega)} = \sum_{p=1}^{\infty} \int_0^{L_1} \|\Psi_f(x,p)\|^2_{\mathbf{R}^2}\, dx$$

In a similar fashion:

$$\left\|\frac{\partial^2}{\partial^2 y}\Psi_f(x,y)\right\|^2_{\mathbf{L}^2(\Omega)} = \sum_{p=1}^{\infty} p^4 \int_0^{L_1} \|\Psi_f(x,p)\|^2_{\mathbf{R}^2}\, dx \qquad \text{(B.2)}$$

We wish to prove that this norm is finite. For separable boundary conditions one can verify that

$$\|F_{fb}^{-1} V_b^0\| < g_2 p$$

and

$$\left\|\left[e^{\Lambda_p x} \vdots 0\right]\right\| < g_3 e^{\mathrm{Re}\lambda_0 x}$$

Combining these bounds gives

$$\|\Psi_f^{\beta}(x,p)\|_{\mathbf{R}^2} < g_4 p^{-1/2} M(p) e^{\mathrm{Re}\lambda_0 x}$$

so that

$$\left\|\frac{\partial^2}{\partial^2 y}\Psi_f^{\beta}(x,y)\right\|^2_{\mathbf{L}^2(\Omega)} < \sum_{p=1}^{\infty} p^3 g_4 M^2(p) \int_0^{L_1} e^{2\mathrm{Re}\lambda_0 x}\, dx$$

or

$$\left\|\frac{\partial^2}{\partial^2 y}\Psi_f^{\beta}(x,y)\right\|^2_{\mathbf{L}^2(\Omega)} < \sum_{p=1}^{\infty} p^2 g_5 M^2(p)$$

since $\lambda_0 < g_6 p$. If we assume that $(\partial/\partial y) z(x,y) \in \mathbf{L}^2(\Omega)$ then

$$\left\|\frac{\partial}{\partial y} z(x,y)\right\|_{\mathbf{L}^2(\Omega)} = \sum_{p=1}^{\infty} p^2 M^2(p) < \infty$$

which shows that $\Psi_f^{\beta}(x,y)$ is twice differentiable in the y-direction. Similarly,

$$\left\|\frac{\partial^2}{\partial^2 y}\Psi_f^{0}(x,y)\right\|_{\mathbf{L}^2(\Omega)} < \sum_{p=1}^{\infty} L_1 p M^2(p) < \infty$$

which implies that $\Psi_f(x,y)$ has a mean-square second derivative in the y-direction. A similar argument shows that $\Psi_b(x,y)$ is also mean-square twice differentiable in the y-direction.

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A fast algorithm for reconstructing images governed by a 2-D Helmholtz equation is presented. The computational complexity is $O(NM\log M)$ or $O(NM^2)$, depending on boundary conditions, where N and M are the number of spatial grid points in the x and y directions, respectively. We show that the smoothing problem is well-posed, and that the sample functions of the smoothed estimate possess smoothness properties consistent with the Helmholtz equation.

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